1 Barometric formula (30 Pts)

1.1 (10 Pts)

The air pressure decreases with increasing height from the earth’s surface. Derive the barometric formula that describes this process with help of the hydrostatic balance equation. (We assume that air can be treated as an ideal gas and that the temperature remains constant with height.)

1.2 (10 Pts)

Now we assume that the temperature decreases linearly with height, starting at $T_{\text{Bottom}} = 298$ K and decreasing with a rate of $a = 0.01$ K/m. How does this change the barometric formula?

1.3 (10 Pts)

What is the total mass of the atmosphere and at which height exactly half of the air mass can be found below this height (The earth radius is $r_E = 6.371 \times 10^6$ m)?
2 Flow between Parallel Plates (30 Pts)

A viscous Newtonian fluid is flowing between two parallel plates located at \( y = 0 \) and \( y = h \) as shown in Figure 1. The top plate at \( y = h \) moves with a constant velocity \( U \) in \( x \)-direction and an external pressure gradient \( \frac{dp}{dx} \) is imposed.

![Coordinate system for flow between parallel plates. (© Kundu)](image)

2.1 (20 Pts)

a) Using the Navier-Stokes equations, compute the flow field for an arbitrary \( U \) and \( p \).

b) Compute the velocity profile \( u(y) \) and sketch it for the following four cases:

1. \( U > 0, \frac{dp}{dx} < 0 \)
2. \( U > 0, \frac{dp}{dx} > 0 \)
3. \( U > 0, \frac{dp}{dx} = 0 \)
4. \( U = 0, \frac{dp}{dx} < 0 \)

How does a fluid parcel in the channel behave as it is advected for these cases?

2.2 (5 Pts)

The special case \( U = 0, \frac{dp}{dx} < 0 \) is termed **plane Poiseuille flow**. Identify the velocity profile \( u(y) \) and compute the magnitude of the shear stress \( \tau \).

2.3 (5 Pts)

The special case \( U > 0, \frac{dp}{dx} = 0 \) is termed **plane Couette flow**. Identify the velocity profile \( u(y) \) and compute the magnitude of the shear stress \( \tau \).
3 Linear waves (30 Pts)

Water waves can be approximated to first order by linear wave theory (cf. Kundu). We consider sinusoidal waves with a surface elevation \( \eta = a \cos (kx - \omega t) \) that propagate in \( x \)-direction only.

The velocity components are:

\[
\begin{align*}
    u &= a \omega \frac{\cosh (k(z + H))}{\sinh(kH)} \cosh(kx - \omega t) \\
    w &= a \omega \frac{\sinh(k(z + H))}{\sinh(kH)} \cosh(kx - \omega t)
\end{align*}
\]

\( u, w \): velocity component in \( x, z \)-direction, \( k \): wave number, \( \omega \): frequency, \( H \): water depth, c.f. Figure 2.

Figure 2: Wave nomenclature (© Kundu)

1. Write down the definition of Streamlines, Path Lines, und Streak Lines.

2. Calculate and sketch the Streamlines, Path Lines, und Streak Lines.