1 Brunt-Väisälä frequency (40 Pts)

1.1 20 Pts

Derive the expression for the Brunt-Väisälä frequency. (Hint: Assume an incompressible fluid with variable density for which the Boussinesq approximation applies. Start with the continuity equation in the correct form and Euler’s equation. There are no density variations along the x- and y- axis only along the z- axis.)
1.2 10 Pts

You are an oceanographer and have just measured the following density profile along the water column in the South Pacific (Antarctic Circumpolar Current):

![Density Profile](image)

Figure 3: Density variation along the water column (sigma-0 denotes $\sigma_0 = \rho - 1000 \text{ kg/m}^3$).

Sketch the corresponding profile of the Brunt-Väisälä frequency that you would expect to find here.

1.3 10 Pts

Estimate the density gradient from the profile above and calculate the maximum Brunt-Väisälä frequency in the profile.

2 Dolphins (20 Pts)

Dolphins can travel at surprisingly high speeds of up to 15 m/s. According to the 'Gray’s Paradoxon' they would therefore need far higher muscle power than their muscles actually provide.

Calculate the highest possible swimming velocities in the cases of laminar and turbulent flow. (Hint: You need the equation for drag resistance. Assume an ellipsoid with a length of $l = 2m$ and a width of $d = l/4$. Muscle power per kg can be assumed to be 0.5 W/kg. The drag coefficients...
are:
\[ c_f = 0.074/Re^{1/5} \] for turbulent flow, and
\[ c_f = 1.328/Re^{1/2} \] for laminar flow.

What does this mean for the dolphin?

3 Divergence and Vorticity (20 Pts)

The Stream function of a flow is defined by:
\[ \frac{\partial \psi}{\partial y} = -u; \frac{\partial \psi}{\partial x} = v \] (1)

The velocity potential is defined by:
\[ \nabla \phi = \vec{u} \] (2)

3.1

Show that:
Every flow that is described by a stream function has zero divergence.
Every flow that is described by a potential has zero vorticity.

3.2

Express:
The vorticity in terms of the stream function.
The divergence in terms of the potential.

3.3

Draw the velocity vectors and streamlines and calculate vorticity and divergence of the following velocity fields:
\[ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}; \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos y \\ \sin x \end{pmatrix} \] (3)