1 Numerical Analysis of the Heat Conduction Equation

1.1 Stability analysis

For the heat conduction equation \( \partial T/\partial t - D \partial^2 T/\partial x^2 \), one of the discretized forms is

\[
-sT_{j+1}^{n+1} + (1 + 2s)T_j^{n+1} - sT_{j-1}^{n+1} = T_j^n
\]  

where \( s = D\Delta t/\Delta x^2 \). Show that this implicit algorithm is always stable.

1.2 Numerical Solution

An insulated rod initially has a temperature of \( T(x,0) = 0^\circ C \) (\( 0 \leq x \leq 1 \)). At \( t=0 \) hot reservoirs \( (T = 100^\circ C) \) are brought into contact with the two ends, \( A(x=0) \) and \( B(x=1) \): \( T(0,t) = T(1,t) = 100^\circ C \). Numerically find the temperature \( T(x,t) \) of any point in the rod. The governing equation of the problem is the heat conduction equation \( \partial T/\partial t - D \partial^2 T/\partial x^2 \). The exact solution to this problem is

\[
T^*(x_j,t_n) = 100 - \sum_{m=1}^{NM} \frac{400}{(2m - 1)\pi} \sin((2m - 1)\pi x_j) \exp(-D(2m - 1)^2\pi^2 t_n)
\]  

where \( NM \) is the number of terms used in the approximation.

1.2.1

Try to solve the problem with the explicit FTCS (forward time, central space) scheme. Use the parameter \( s = D\Delta t/\Delta x^2 = 0.5 \) and \( 0.6 \) to test the stability of the scheme.

1.2.2

Solve the problem with a stable explicit or implicit scheme. Test the rate of convergence numerically, using the error at \( x=0.5 \).